A TYPE OF THE LEFSCHETZ HYPERPLANE SECTION THEOREM ON $\mathbb{Q}$-FANO 3-FOLDS WITH PICARD NUMBER ONE AND $\frac{1}{2}(1,1,1)$-SINGULARITIES

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Dedicated to Prof. Daniel Burns on the occasion of his 65th birthday

Abstract. We prove a type of the Lefschetz hyperplane section theorem on $\mathbb{Q}$-Fano 3-folds with Picard number one and $\frac{1}{2}(1,1,1)$-singularities by using some degeneration method. As a byproduct, we obtain a new example of a Calabi–Yau 3-fold X with Picard number one whose invariants are

$$(H_X^3,\, c_2(X)\cdot H_X,\, e(X)) = (8, 44, -88),$$

where H_X, $e(X)$ and $c_2(X)$ are an ample generator of $\mathrm{Pic}(X)$, the topological Euler characteristic number and the second Chern class of X respectively.

0. Introduction

We work over the complex number field. Let us start by the definition of $\mathbb{Q}$-Fano variety.

Definition 0.1. *A projective variety Y is called a* $\mathbb{Q}$-Fano variety *if Y has only terminal singularities and $-K_Y$ is ample.*

As the smooth Fano 3-folds, $\mathbb{Q}$-Fano varieties with Picard number one form an important class among general ones. In this note, we investigate a type of the Lefschetz hyperplane section theorem on certain $\mathbb{Q}$-Fano 3-folds. Let Y be a $\mathbb{Q}$-Fano 3-fold with Picard number one and only cyclic singularities of type $\frac{1}{2}(1,1,1)$. Such $\mathbb{Q}$-Fano 3-folds have been extensively investigated in [Sa1, Sa2], [CaFl] and [Ta1, Ta2]. We assume the following:

Condition 0.2.

(1) *The divisor class group* $\mathrm{Cl}(Y)$ *of Y is generated a single element h.*
(2) *$-2K_Y$ is very ample.*
(3) *The linear system $|-K_Y|$ has a member D that is smooth outside* $\mathrm{Sing}(Y)$ *and has singularities of type $\frac{1}{2}(1,1)$ at* $\mathrm{Sing}(Y)$.

Note that all the examples of $\mathbb{Q}$-Fano 3-folds classified by H. Takagi satisfy this condition (Corollary 3.4 in [Ta2]).

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Let Y_n be a Weil divisor of Y such that $Y_n \sim nh$. If n is odd, then Y_n can not be smooth and at best it has singularities of type $\frac{1}{2}(1,1)$ at $\mathrm{Sing}(Y)$ – call such one quasi-smooth. If n is even, Y_n can be smooth. The purpose of this note is to prove the following type of the Lefschetz hyperplane section theorem on Y.

Theorem 0.3. *Let Y be a $\mathbb{Q}$-Fano 3-fold with Picard number one and only $\frac{1}{2}(1,1,1)$-singularities, satisfying Condition 0.2, and Y_n be the smooth or quasi-smooth divisor on Y defined as above. Then the homomorphism of divisor class groups*

$$\mathrm{Cl}(Y) \to \mathrm{Cl}(Y_n)_f$$

is injective and primitive (i.e. the cokernel is torsion-free). For an abelian group A, we set $A_f := A/(\text{torsion part of } A)$.

As a byproduct of the proof, we will be able to calculate some invariants of a Calabi–Yau 3-fold, which was constructed in [Le1] but whose invariants could not be obtained there. It turns out that it is a new example of Calabi–Yau 3-fold with Picard number one (§3). Its invariants are

$$(H_X^3, c_2(X) \cdot H_X, e(X)) = (8, 44, -88),$$

where H_X, $e(X)$ and $c_2(X)$ are an ample generator of $\mathrm{Pic}(X)$, the topological Euler characteristic number and the second Chern class of X respectively.

The idea of the proof is to use a *smooth* double covering X over Y. Firstly we determine the Picard group of X by some degeneration method. Then we can calculate invariants of the preimage of Y_n by the ordinary Lefschetz hyperplane section theorem on X. With this we finally obtain the invariants of Y_n.

In §1, we briefly introduce some degeneration method from [Le2]. §2 is devoted to calculation of the Picard group of the blow-up of Y at its singularities. In §3, we apply the degeneration method to calculate invariants of X and prove the main theorem.

1. Degeneration and integral cohomology groups

We summarize some results of [Le2] that will be used in this note.

Let $\phi : W \to \bar{\Delta}$ be a proper map from a smooth $(n+1)$-fold W with boundary onto a closed disk $\bar{\Delta} = \{t \in \mathbb{C} \mid \|t\| \leq 1\}$ such that the fibers $W_t = \phi^{-1}(t)$ are connected Kähler n-folds for every $t \neq 0$ (generic) and the central fiber $W_0 = \phi^{-1}(0) = \bigcup_\alpha V_\alpha$ is a normal crossing of n-folds. We denote the general fiber by W_t. The condition, $t \neq 0$, is assumed in this notation. We call such a map ϕ (or simply the total space W) a semi-stable degeneration of W_t and we say that W_0 is smoothable to W_t with the smooth total space. Let $V_{ij} = V_i \cap V_j$, $V_{ijk} = V_i \cap V_j \cap V_k$ and etc. Consider the exact sequence

$$0 \to \mathbb{Z}_{W_0} \to \bigoplus_\alpha \mathbb{Z}_{V_\alpha} \to \bigoplus_{i<j} \mathbb{Z}_{V_{ij}} \xrightarrow{\tau} \bigoplus_{i<j<k} \mathbb{Z}_{V_{ijk}} \to \cdots$$

We introduce a short exact sequence,

$$0 \to \mathbb{Z}_{W_0} \to \bigoplus_\alpha \mathbb{Z}_{V_\alpha} \to \ker(\tau) \to 0$$

to give an exact sequence

$$\cdots \to H^m(W_0, \mathbb{Z}) \xrightarrow{\psi_m} \bigoplus_\alpha H^m(V_\alpha, \mathbb{Z}) \to H^m(\ker(\tau)) \to \cdots \tag{1.1}$$

Note that $\psi_m = \bigoplus_\alpha j_\alpha^*|_{H^m(W_0,\mathbb{Z})}$, where $j_\alpha : V_\alpha \hookrightarrow W_0$ is the inclusion. Let

$$G^m(W_0, \mathbb{Z}) = \mathrm{im}(\psi_m) \subset \bigoplus_\alpha H^m(V_\alpha, \mathbb{Z}).$$

Consider non-negative integers, $q_1, \cdots, q_k$ such that

$$q_1 + \cdots + q_k = n.$$

Then there are multilinear maps, defined by the cup-product:

$$H^{2q_1}(W_t, \mathbb{Z}) \times \cdots \times H^{2q_k}(W_t, \mathbb{Z}) \to \mathbb{Z},$$

$$H^{2q_1}(V_\alpha, \mathbb{Z}) \times \cdots \times H^{2q_k}(V_\alpha, \mathbb{Z}) \to \mathbb{Z}.$$

The latter one induces a multilinear map

$$\bigoplus_{\alpha_1} H^{2q_1}(V_{\alpha_1}, \mathbb{Z}) \times \cdots \times \bigoplus_{\alpha_k} H^{2q_k}(V_{\alpha_k}, \mathbb{Z}) \to \mathbb{Z},$$

setting the mixed terms to be equal to zero. By restricting, we can define a multilinear map

$$G^{2q_1}(W_0, \mathbb{Z}) \times \cdots \times G^{2q_k}(W_0, \mathbb{Z}) \to \mathbb{Z}.$$

Note that this multilinear map define a cup product on $G^2(W_0, \mathbb{Z})$.

Let $h^{p,q}(W_t) = \dim H^{p,q}(W_t)$ and $h^i(W_t) = \dim H^i(W_t)$. We will need the following simplified results ([Le2], Theorem 6.5):

Theorem 1.1. *Suppose that $h^{2,0}(W_t) = 0$. Let $a_1, a_2, \cdots, a_k \in G^2(W_0, \mathbb{Z})$ with $k = h^2(W_t)$ and $b_1, b_2, \cdots, b_k \in G^{n-2}(W_0, \mathbb{Z})$ such that the intersection matrix of cup products:*

$$(a_i \cdot b_j)$$

is unimodular. Then the sublattice $\langle a_1, \cdots, a_m\rangle$ of $G^2(W_0, \mathbb{Z})$ is isomorphic to $H^2(W_t, \mathbb{Z})_f$ with the cup product preserved.

2. Picard group of some blow-up

From now on, the variety Y is a $\mathbb{Q}$-Fano 3-fold as in Theorem 0.3. Under (1) in Condition 0.2, the class group $\mathrm{Cl}(Y)$ is generated by an element h with $h^3 > 0$. Let $f : V_1 \to Y$ be the blow up at $\mathrm{Sing}(Y) = \{p_1, \cdots, p_N\}$ and e_i's be the exceptional divisors over p_i's. Let B be a prime divisor on Y. Then we have

$$f^*(B) = \widetilde{B} + \sum_i \frac{1}{2} q_i e_i,$$

for some nonnegative integer q_i, where $\widetilde{B}$ is the proper transform of B. We define the number q_i as the multiplicity $\mathrm{mult}_p(B)$ of B at $p_i \in \mathrm{Sing}(Y)$. It is easy to see that B is Cartier at p_i if and only if $\mathrm{mult}_{p_i}(B)$ is even. The purpose of this section is to show:

Lemma 2.1. *The Picard group of V_1 is*

$$\mathrm{Pic}(V_1) = \langle f^*(h) - \frac{1}{2}\sum_i e_i, e_1, \cdots, e_N \rangle.$$

Proof. Since $\dim Y$ is odd, K_Y is *not* Cartier at $p_i \in \mathrm{Sing}(Y)$. Let $K_Y \sim -rh$ for some integer r. Note that r is odd since otherwise K_Y would be Cartier. Note that $h \sim (r+1)h + K_Y$ and $(r+1)h$ is Cartier. So h is not Cartier at any point p_i.

Note that $f^*(K_Y) + \frac{1}{2}\sum_i e_i = K_{V_1}$. So the following rational combination of divisors

$$\begin{aligned} f^*(h) - \frac{1}{2}\sum_i e_i &\sim f^*((r+1)h) + f^*(K_Y) + \frac{1}{2}\sum_i e_i - \sum_i e_i \\ &\sim f^*((r+1)h) + K_{V_1} - \sum_i e_i \end{aligned}$$

is actually a divisor class since $(r+1)$ is even. Consider a subgroup

$$G := \langle f^*(h) - \frac{1}{2}\sum_i e_i,\ e_1, \cdots, e_N \rangle$$

of $\mathrm{Pic}(Y)$. Let L be a prime divisor on V_1 such that $e_i \not\subset L$ for any i. Let q_i be the degree $L|_{e_i}$ in e_i. Then we have

$$L = f^*(\check{L}) - \sum_i \frac{1}{2} q_i e_i,$$

where $\check{L} = f(L)$. Note that $q_i = \mathrm{mult}_{p_i}(\check{L})$. We can set $\check{L} \sim kh$ for some integer k. Note that the following are equivalent:

(1) q_i is even,
(2) $\check{L}$ is Cartier at p_i,
(3) kh is Cartier at p_i,
(4) k is even.

So we have

$$k \equiv q_i (\mathrm{mod} 2)$$

for any i. Then

$$\begin{aligned} L &= f^*(\check{L}) - \sum_i \frac{1}{2} q_i e_i \\ &\sim k f^*(h) - \sum_i \frac{1}{2} q_i e_i \\ &= k\left(f^*(h) - \frac{1}{2}\sum_i e_i \right) + \sum_i \frac{1}{2}(q_i - k) e_i. \end{aligned}$$

Since $(q_i - k)$'s are even, $\frac{q_i-k}{2}$'s are integers and we have $L \in G$. A divisor is linear equivalent to a linear combination of prime divisors, so any divisor class belongs to G. Therefore we have $\mathrm{Pic}(V_1) \subset G$. □

Remark 2.2. If $\mathrm{Cl}(Y)$ has non-zero torsion part or its rank is higher than one, the computation of $\mathrm{Pic}(V_1)$ is non-trivial and depends on the detailed geometry of Y.

Since $h^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < 3$, we have

$$H^2(V_1, \mathbb{Z}) \simeq \mathrm{Pic}(V_1) = \langle f^*(h) - \frac{1}{2}\sum_i e_i, e_1, \cdots, e_N \rangle.$$

By Poincaré duality, there are classes $m_0, m_1, \cdots, m_N$ in $H^4(V_1, \mathbb{Z})$ whose cup product matrix with $f^*(h) - \frac{1}{2}\sum_i e_i, e_1, \cdots, e_N$ is the $(N+1)\times(N+1)$ identity matrix.

3. The proof

Under (2) in Condition 0.2, one can find a smooth surface S in $|-2K_Y|$ such that $S \cap Y_n$ and $C := S \cap D$ are are all smooth curves. Then there is a Calabi–Yau 3-fold X of Picard number one (Theorem 2.1 of [Le1]) that is a double covering over Y with the branch locus

$$S \cup \mathrm{Sing}(Y)$$

([Le1], Theorem 1.1).

Now we use a semistable degeneration $W \to \Delta$ of X, which we constructed in the proof of Theorem 2.1 of [Le1] (p.539 - 540). The generic fiber W_t is a deformation of X.

To describe the central fiber W_0, we introduce some notation.

- Let V_1 be the blow-up of Y at $\mathrm{Sing}(Y)$ as before. Take surfaces S, D as above. Note that the proper transform $\widetilde{D}$ is a smooth $K3$ surface.
- Let $g : V_2 \to Y$ be the blow-up at $\mathrm{Sing}(Y)$ and $C := S \cap D$, and $f_1, \ldots, f_N$ the g-exceptional divisors over $\mathrm{Sing}(Y)$. Let $\widetilde{D'}$ be the proper transform of D.
- Let $E_1, \ldots, E_N$ be copies of $\mathbb{P}^3$'s.

Then $W_0 = V_1 \cup V_2 \cup E_1 \cup \cdots \cup E_N$, where $V_1 \cap V_2 = \widetilde{D} = \widetilde{D'}$, $V_1 \cap E_i = e_i$, $V_2 \cap E_i = f_i$, $e_i \neq f_i$ $(1 \le i \le N)$, and $E_i \cap E_j = \emptyset$ $(1 \le i < j \le N)$. e_i

and f_i are planes of E_i. Note that e_i and f_i were denoted by H_{1i} and H_{2i} respectively in [Le1].

For simplicity, let $\mathrm{Sing}(Y) = \{p\}$ be composed of a single point ($N = 1$). The proof for general cases is similar. Let $q = 2$ or 4 and abbreviate $H^*(*,\mathbb{Z}), G^*(*,\mathbb{Z})$ as in §1 by $H^*(*), G^*(*)$ respectively. Since

$$H^{q-1}(\widetilde{D}) = H^{q-1}(e_1 \cap f_1) = 0,$$

the Mayer–Vietories sequences are:

$$0 \to H^q(V_1 \cup V_2) \to H^q(V_1) \oplus H^q(V_2) \to H^q(\widetilde{D})$$

and

$$0 \to H^q(e_1 \cup f_1) \to H^q(e_1) \oplus H^q(f_1) \to H^q(e_1 \cap f_1).$$

So we can recognize

$$H^q(V_1 \cup V_2) \text{ and } H^q(e_1 \cup f_1)$$

as sublattices of

$$H^q(V_1) \oplus H^q(V_2) \text{ and } H^q(e_1) \oplus H^q(f_1)$$

respectively. Let us denote them by

$$H^q(V_1) \mathbin{\widehat{\oplus}} H^q(V_2) \text{ and } H^q(e_1) \mathbin{\widehat{\oplus}} H^q(f_1)$$

respectively.

Consider the following commutative diagram:

$$\begin{tikzcd}
H^q(W_0) \arrow[rr, "\psi_2"] \arrow[d, dashed] & & H^q(V_1) \oplus H^q(V_2) \oplus H^q(E_1) \\
H^q(V_1 \cup V_2) \oplus H^q(E_1) \arrow[urr] \arrow[r, equal] \arrow[d, dashed] & \left(H^q(V_1) \mathbin{\widehat{\oplus}} H^q(V_2)\right) \oplus H^q(E_1) \arrow[r, phantom] & \\
H^q(e_1 \cup f_1) \arrow[r, equal] & H^q(e_1) \mathbin{\widehat{\oplus}} H^q(f_1) &
\end{tikzcd}$$

where $i: \left(H^q(V_1) \mathbin{\widehat{\oplus}} H^q(V_2)\right) \oplus H^q(E_1) \hookrightarrow H^q(V_1) \oplus H^q(V_2) \oplus H^q(E_1)$ and $\nu: \left(H^q(V_1) \mathbin{\widehat{\oplus}} H^q(V_2)\right) \oplus H^q(E_1) \to H^q(e_1) \mathbin{\widehat{\oplus}} H^q(f_1)$ are the vertical arrows on the right,

where the sequence of dotted arrows is exact, coming from the Mayer–Vietories sequences of the pair $V_1 \cup V_2$, E_1, i is the inclusion map, and the map

$$\nu : \left(H^q(V_1) \mathbin{\widehat{\oplus}} H^q(V_2)\right) \oplus H^q(E_1) \to H^q(e_1) \mathbin{\widehat{\oplus}} H^q(f_1)$$

acts as follows:

$$(l_1, l_2, l_3) \mapsto (l_1|_{e_1} - l_3|_{e_1}, l_2|_{f_1} - l_3|_{f_1}).$$

Therefore we have

$$
\begin{aligned}
G^q(W_0) &= \ker \nu \\
&= \{(l_1, l_2, l_3) \in \left(\widehat{H^q(V_1) \oplus H^q(V_2)}\right) \oplus H^q(E_1) \mid l_1|_{e_1} = l_3|_{e_1}, l_2|_{f_1} = l_3|_{f_1}\} \\
&= \{(l_1, l_2, l_3) \in H^q(V_1) \oplus H^q(V_2) \oplus H^q(E_1) \mid l_1|_{e_1} = l_3|_{e_1}, l_2|_{f_1} = l_3|_{f_1}, l_1|_{\widetilde{D}} = l_2|_{\widetilde{D}}\} \\
&= \{(l_1, l_2, l_3) \in H^q(V_1) \oplus H^q(V_2) \oplus H^q(E_1) \mid l_1, l_2, l_3 \text{ are compatible.}\},
\end{aligned}
$$

where 'compatible' means that the restrictions of the classes l_1, l_2, l_3 to the intersections of V_1, V_2, E_1 coincide. In the general case that $\mathrm{Sing}(Y)$ consists of multiple points ($N \geq 1$), we have

$$
\begin{aligned}
G^q(W_0) = \{(l_1, \cdots, l_{N+2}) \in H^q(V_1) \oplus H^q(V_2) \oplus H^q(E_1) \oplus \cdots \oplus H^q(E_N) \mid \\
l_1, \cdots, l_{N+2} \text{ are compatible.}\}
\end{aligned}
$$

With this, one can easily verify that

$$
H := \left(f^*(h) - \frac{1}{2}\sum_i e_i, g^*(f^*(h) - \frac{1}{2}\sum_i f_i), e_1, \cdots, e_N\right)
$$

and

$$
H' := (m_0, dF, 0, \cdots, 0)
$$

belong to $G^2(W_0)$ and $G^4(W_0)$ respectively, where $F \in H^4(V_2)$ is a fiber class of the blow up $g : V_2 \to Y$ over a point in C and $d = m_0 \cdot \widetilde{D}$. See the end of §2 for the definition of m_0. Finally we get the following lemma:

Lemma 3.1. *The lattice $\langle H \rangle$ is isomorphic to $H^2(W_t, \mathbb{Z})_f$ with the cup product preserved.*

Proof. Note that

$$
H \cdot H' = \left(f^*(h) - \frac{1}{2}\sum_i e_i\right) \cdot m_0 + g^*(f^*(h) - \frac{1}{2}\sum_i f_i) \cdot dF + 0 + \cdots + 0 = 1 + 0 = 1.
$$

Since $h^2(W_t) = 1$ and $h^{2,0}(W_t) = 0$, we are done by Theorem 1.1. $\square$

Note that $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z}) \simeq H^2(W_t, \mathbb{Z})$ and $\mathrm{rkPic}(X) = 1$. Let $\mathrm{Pic}(X)_f = \langle H_X \rangle$, where H_X be an ample generator. Then we have

$$
\begin{aligned}
H_X^3 = H^3 &= \left(f^*(h) - \frac{1}{2}\sum_i e_i\right)^3 + g^*(f^*(h) - \frac{1}{2}\sum_i f_i)^3 + e_1^3 + \cdots + e_N^3 \\
&= \left(h^3 - \frac{1}{8}\sum_i e_i^3\right) + \left(h^3 - \frac{1}{8}\sum_i f_i^3\right) + 1 + \cdots + 1 \\
&= \left(h^3 - \frac{1}{8}\sum_i 4\right) + \left(h^3 - \frac{1}{8}\sum_i 4\right) + N \\
&= 2h^3 - N + N \\
&= 2h^3.
\end{aligned}
$$

Theorem 3.2. *The following map:*

$$\pi^* : \mathrm{Cl}(Y) \to \mathrm{Pic}(X)_f$$

is a bijection.

Proof. For some positive integer a, $\pi^*(h) = aH_X$. Note that $a^3H_X^3 = 2h^3$ and $H_X^3 = 2h^3$. So we have $a = 1$ because H_X^3 and h^3 are positive. Therefore the map is bijective. □

Now we attain the goal of this note.

Corollary 3.3 (Theorem 0.2)**.** *The following map of lattices*

$$\mathrm{Cl}(Y) \to \mathrm{Cl}(Y_n)_f$$

is injective and primitive.

Proof. Note that $X_n := \pi^{-1}(Y_n)$ is an ample divisor of X. Then the induced map $\pi|_{X_n} : X_n \to Y_n$ is a double covering, branched over $Y_n \cap S$ if n is even and $(Y_n \cap S) \cup \mathrm{Sing}(X)$ if n is odd. Let $i_Y : Y_n \hookrightarrow Y$ and $i_X : X_n \hookrightarrow X$ be the inclusion maps. By the Lefschetz hyperplane section theorem, the map of lattices

$$i_X^* : \mathrm{Pic}(X)_f \to \mathrm{Pic}(X_n)_f$$

is injective and primitive. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Pic}(X_n)_f & \mathrm{Pic}(X)_f \arrow[l, "i_X^*"'] \\
\mathrm{Cl}(Y_n)_f \arrow[u, "\varsigma"] & \mathrm{Cl}(Y) \arrow[l, "i_Y^*"] \arrow[u, "\pi^*"']
\end{tikzcd}$$

where $\varsigma = (\pi|_{X_n})^* : \mathrm{Cl}(Y_n)_f \to \mathrm{Pic}(X_n)_f$. We showed that the map $\pi^* : \mathrm{Cl}(Y) \to \mathrm{Pic}(X)_f$ is injective and primitive. So $i_X^* \circ \pi^*$ is also injective and primitive. Therefore the map $i_Y^* : \mathrm{Cl}(Y) \to \mathrm{Cl}(Y_n)_f$ should be injective and primitive. □

The arguments can be generalized to the case of odd dimensional $\mathbb{Q}$-Fano n-folds with singularities of type $\frac{1}{2}(1, \cdots, 1)$ and other similar conditions.

Now we can remove the condition of the exceptional case $(-K_Y^3, N) = (4, 4)$ in Theorem 3.2 of [Le1]. The critical part of the proof of Theorem 3.2 of [Le1] is to show that $a = 1$ as in the above proof. In [Le1], we relied on the Riemann–Roch theorem and divisibility of some intersection numbers, which is why the numbers $(-K_Y^3, N)$ mattered there. Now we can get the invariants of the Calabi–Yau double cover of Takagi's $\mathbb{Q}$-Fano 3-fold of $(-K_Y^3, N) = (4, 4)$. The invariants are

$$(H_X^3, c_2(X) \cdot H_X, e(X)) = (8, 44, -88),$$

where $e(X)$ and $c_2(X)$ are the topological Euler characteristic number and the second Chern class of X respectively. Those invariants verify that it is a new example of Calabi–Yau 3-folds with Picard number one, which play

an important role in the moduli spaces of all Calabi–Yau 3-folds ([Gr]). See Table 1 in [EnSt] for a list of some known ones.

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References

[CaFl] F. Campana and H. Flenner, *Projective threefolds containing a smooth rational surface with ample normal bundle*, J. reine angew. Math **440** (1993), 77–98.

[EnSt] C. van Enckevort and D. van Straten, *Monodromy calculations of fourth order equations of Calabi–Yau type, Mirror symmetry. V*, 539–559, AMS/IP Stud. Adv. Math., **38**, Amer. Math. Soc., Providence, RI, 2006.

[Gr] M. Gross, *Primitive Calabi–Yau threefolds*, J. Diff. Geom. **45** (1997), 288–318.

[Le1] N.-H. Lee, *Calabi–Yau coverings over some singular varieties and new Calabi–Yau 3-folds with Picard number one*, Manuscripta Math. **125** (2008), no. 4, 531–547.

[Le2] N.-H. Lee, *Calabi–Yau construction by smoothing normal crossing varieties*, Int. J. of Math. **21** (2010), no. 6, 701–725.

[Sa1] T. Sano, *Classification of non-Gorenstein $\mathbb{Q}$-Fano d-folds of Fano index greater than* $d-2$, Nagoya Math. J. **142** (1996), 133-143.

[Sa2] T. Sano, *On classification of non-Gorenstein Q-Fano* 3*-folds of Fano index* 1, J. Math. Soc. Japan **47** (1995), no. 2, 369–380.

[Ta1] H. Takagi, *On classification of $\mathbb{Q}$-Fano* 3*-folds of Gorenstein index 2. I*, Nagoya Math. J. **167** (2002), 117-155.

[Ta2] H. Takagi, *On classification of $\mathbb{Q}$-Fano* 3*-folds of Gorenstein index 2. II*, Nagoya Math. J. **167** (2002), 157-216.

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